Project 2: Gravitational collapse in the spherically symmetric Einstein-Klein-Gordon system

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1 Introduction

In this project we will study the dynamics of a self-gravitating scalar field; i.e. in contrast to project 1 we will now solve the full Einstein equations (in spherical symmetry) with the scalar field sourcing all the non-trivial geometry that unfolds. The key physics we would like to explore is the process of gravitational collapse and black hole formation. Topics in general relativity that we will encounter include the ADM space-plus-time decomposition, issues of choosing coordinates, constraint and evolution equations, trapped surfaces, apparent horizons black holes and singularities. From a computational standpoint we will solve coupled, nonlinear hyperbolic and elliptic partial differential equations. Time permitting, extensions include a study of critical phenomena at the threshold of black hole formation, and the ideas of adaptive mesh refinement, multigrid solution of elliptic equations and parallelization.

1.1 Units and Conventions

We adopt the same conventions, units and notation as with project 1.

2 Problem specification

We will solve the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

for the components of the metric tensor $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

coupled to a massless scalar field Φ with stress-energy-momentum tensor

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \nabla_\gamma \Phi \nabla^\gamma \Phi. \tag{3}$$

The scalar field satisfies the Klein-Gordon equation

$$\nabla_\gamma \nabla^\gamma \Phi = 0 \tag{4}$$

We will deconstruct these equations using the ADM (Arnowitt-Deser-Minser) (or "3+1" or "space plus time") decomposition of the geometry. The 4-dimensional spacetime line element (2) is written as

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{5}$$

where α is the *lapse function*, β^i the shift vector, and h_{ij} the 3-dimensional spatial metric. Though ultimately we will not directly use it, an important geometric object in the ADM split is the *extrinsic curvature tensor* $K_{\mu\nu}$, defined as

$$K_{\mu\nu} \equiv - \perp_\mu{}^\gamma \perp_\nu{}^\delta \nabla_\gamma n_\delta, \tag{6}$$

where n^μ is the unit, time-like vector field normal to t = const surfaces, and $\perp_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$ is the *projection tensor* projecting 4-dimensional tensorial objects onto the t = const slice. We now restrict to spherical symmetry, using spherical-polar coordinates (r, θ, ϕ), and choose the spatial metric to be in *isotropic* (conformally flat) form. In spherical symmetry only the $r - component$ of the shift vector can be non-zero, and we will call that β for simplicity. This gives a line element

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)dt^2 + 2\psi^4\beta dr dt + \psi^4\left[dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], \quad (7)$$

where all variables $\alpha(r,t), \beta(r,t), \psi(r,t)$ (and the scalar field $\Phi(r,t)$) depend on r and t. There is still have one degree of freedom freedom unspecified in the this form of the metric, and we will fix it by imposing *maximal slicing*, which demands that the trace of the extrinsic curvature tensor $K_\mu{}^\mu$ vanish for all time. One can show that this condition is equivalent to demanding that the fractional change of the local volume element (which here is $\psi^6 r^2 \sin(\theta)$) is zero in the direction normal to the hypersurface. Note that this condition does not simplify the metric, though it does (together with spherical symmetry) allow us to completely eliminate the extrinsic curvature tensor from all the equations.

2.1 Equations

Here we simply present the ADM form of the Einstein-Klein-Gordon equations, adapted to a spherically symmetric spacetime in maximal isotropic coordinates as described above, without derivation. In the equations below, an overdot ($\dot{}$) denotes partial differentiation with respect to t, and a prime ($'$) partial differentiation with respect to r.

First, we will reduce the Klein-Gordon equation to first order form by introducing the auxiliary variables ξ and Π (not to be confused with the number π):

$$\xi(r,t) \equiv \Phi' \quad (8)$$

$$\Pi(r,t) \equiv \frac{\psi^2}{\alpha}\left(\dot{\Phi} - \beta\xi\right) \quad (9)$$

In terms of these variables, the hyperbolic Klein-Gordon equation is

$$\dot{\Pi} - \frac{1}{r^2\psi^4}\left[r^2\psi^4\left(\beta\Pi + \frac{\alpha\xi}{\psi^2}\right)\right]' + \frac{2}{3}\Pi\left[\beta' + \frac{2\beta}{r}\left(1 + \frac{3r\psi'}{\psi}\right)\right] = 0 \quad (10)$$

The hyperbolic evolution equation for ξ can be derived from (8,9)

$$\dot{\xi} - (\alpha\Pi/\psi^2 + \beta\xi)' = 0. \quad (11)$$

The *Hamiltonian constraint* equation, which we treat as an elliptic equation for the conformal factor ψ, is

$$\psi'' + \psi'\left[\frac{2}{r}\right] + \frac{\psi^5}{12}\left[\frac{1}{\alpha}\left(\beta' - \frac{\beta}{r}\right)\right]^2 + \pi\psi\left[\xi^2 + \Pi^2\right] = 0. \quad (12)$$

The *momentum constraint* equation, an elliptic equation for β, is

$$\beta'' + \left(\beta' - \frac{\beta}{r}\right)\left[\frac{2}{r} + \frac{6\psi'}{\psi} - \frac{\alpha'}{\alpha}\right] + \frac{12\pi\alpha\xi\Pi}{\psi^2} = 0 \quad (13)$$

The maximal slicing condition gives an elliptic equation for α

$$\alpha'' + \alpha'\left[\frac{2}{r} + \frac{2\psi'}{\psi}\right] - \alpha^{-1}\left[\frac{2\psi^4}{3}\left(\beta' - \frac{\beta}{r}\right)^2\right] - 8\pi\alpha\Pi^2 = 0. \quad (14)$$

Equations (10-14) form a closed system that we can solve for the unknowns in the problem:$\{\Pi, \xi, \psi, \beta, \alpha\}$. An evolution scheme based on this system is called *fully constrained*. However, ψ also satisfies the following hyperbolic evolution equation (from the maximal slicing condition):

$$\dot{\psi} - \beta\left(\frac{\psi}{3r} + \psi'\right) - \frac{\psi\beta'}{6} = 0. \quad (15)$$

After the initial time (when the Hamiltonian constraint must be solved to provide consistent initial data), the above can be used in lieu of (12) to solve for ψ.

3 Problems

NOTE: For all questions where an answer about a numerical solution is required, report the answer with an error estimate based on convergence testing and the Richardson expansion. In other words, the simulation should be repeated with (at least) 3 resolutions: low (level 0), medium (level 1, i.e. 1/2 the mesh spacing of level 0) and high (level 2, 1/4 the the mesh spacing). Three resolutions are needed to demonstrate you are in the convergent regime, the answer will be the result from the highest resolution, and the medium and highest resolution results can be used to provide an estimate of the solution error. Some experimentation may be required to find an acceptable set of resolutions. On one hand, you want the simulations to run as quickly as possible for fast turn-around time, which argues for lower resolution. However you need at least enough resolution so that you are in the convergent regime where the Richardson expansion is valid, and the minimum resolution for this will depend on the particular class of initial conditions you are evolving as well as the questions you are trying to answer.

3.1 Problem 1. Regularity conditions at $r = 0$

Spherical polar coordinates are singular at the origin, which introduces terms that go as $1/r$ or $1/r^2$ in the equations above. However, we will demand that all solutions $\{\Pi, \xi, \psi, \beta, \alpha\}$ be *regular* and *finite* at $r = 0$, which requires that we impose *regularity conditions* on our functions at $r = 0$. Show that ψ, α and Π need *Neumann* conditions, and β and ξ *Dirichlet* conditions at $r = 0$ for regularity:

$$\psi'(r = 0, t) = 0 \tag{16}$$
$$\alpha'(r = 0, t) = 0 \tag{17}$$
$$\Pi'(r = 0, t) = 0 \tag{18}$$
$$\beta(r = 0, t) = 0 \tag{19}$$
$$\xi(r = 0, t) = 0 \tag{20}$$

3.2 Problem 2. Outer boundary conditions

At the outer boundary of our computational domain, $r = R$, derive a set of boundary conditions for the metric variables that are consistent, to order $1/r^2$, with asymptotic flatness; i.e. we require

$$\psi(r = R, t) = 1 + a_0(t)/r + O(1/r^2), \tag{21}$$
$$\alpha(r = R, t) = 1 + b_0(t)/r + O(1/r^2), \tag{22}$$
$$\beta(r = R, t) = c_0(t)/r + O(1/r^2), \tag{23}$$

where a_0, b_0 and c_0 are *unknown* functions of time. For the matter, derive outer boundary conditions for Π and ξ that are consistent, to $O(1/r^2)$, with no incoming radiation at $r = R$.

Hint: to $O(1/r^2)$ the conditions are a bit simpler than the corresponding conditions on the matter for project 1. The reason is here the metric is dynamic, and to keep things as simple as possible we would like to avoid boundary conditions that couple the matter variables to the metric variables. This comes at the expense of having a less accurate approximation to the no-incoming-radiation condition at a given radius.

3.3 Problem 3. Solve the Klein Gordon equation

When constructing a numerical code solving a rather complicated system of coupled, non-linear equations as we have here, it makes sense to build toward the final code in as many small steps as possible, fully testing each addition before moving onto the next step. This and the next set of problems are designed to help you along in that regard.

Write a code in RNPL to solve the Klein-Gordon equation in the form (10,11), together with the boundary conditions you derived in Problems 1 and 2, using second order accurate Crank-Nicholson-style finite difference techniques as used in Project 1. Also integrate (9) to solve for the field $\phi(r, t)$. Implement initial conditions representing an (approximate) ingoing Gaussian pulse centered at r_0, with characteristic width δ in r and an amplitude A. At this stage, define the geometry variables and include them in the Klein-Gordon equation, but initialize them to a Minkowski background. When possible re-use code segments from Project 1. Test the code to make sure it is second-order convergent.

Hints: to avoid later complications when solving the initial conditions for the full system of equations, derive the ingoing conditions for the scalar field purely in terms of the scalar field variables ξ and Π and the coordinate position r. Kreiss Oliger (KO) dissipation will be necessary, and in particular dissipation stencils that can be applied at $r = \Delta r$ may need to be added for stability near the origin. At $r = \Delta r$ use the usual centered form for KO stencil, but as appropriate reflect the stencil through $r = 0$ taking the even or odd character of the particular variable that it is applied to in consideration.

3.4 Problem 4. Solve the Elliptic Equations

RNPL can only efficiently solve hyperbolic equations (at present). Therefore we need to write custom code to solve the elliptic equations in our system, and link that into RNPL (using the same mechanism we used to compute the energy of the field in Project 1). The procedure we will use is in fact quite similar to what RNPL does, namely a *one step Newton-Gauss-Seidel relaxation*, iterated until convergence; the difference is we will simultaneous solve each iteration of the linearized equations over the entire domain using `LAPACK`'s banded matrix solver. The method is described in Appendix A, and a sample stand-alone code solving a model non-linear elliptic equation will be provided to demonstrate the technique.

Add a routine to solve the three elliptic equations (12,13,14), called from RNPL. During each call, solve the set of three equations to within a specified tolerance. Once you are able to get a solution to the coupled system of elliptic equations, as a test (and for future evolutions) compute a *mass aspect* $m(r,t)$ for the spacetime. Since we are in spherical symmetry we can unambiguously define the total gravitational mass $m(r,t)$ of the spacetime within a radius r. The scalar function $R(r,t) = r\psi(r,t)^2$ is the areal radius of an $r =$ const. sphere. From the Schwarzschild form of the metric line element one can easily see that

$$\nabla_\alpha R \; \nabla_\beta R \; g^{\alpha\beta} = 1 - \frac{2M}{R}, \tag{24}$$

where M is the mass of the corresponding black hole. Since the the above is a spacetime scalar, it is a valid relation in any coordinate system. From this we *define* the mass aspect $m(r,t)$ to be

$$m(r,t) = \frac{R}{2}\left(1 - \nabla_\alpha R \; \nabla_\beta R \; g^{\alpha\beta}\right) \tag{25}$$

By Birkhoff's theorem, $m(r,t)$ will equal the total mass M of the spacetime when evaluated at locations r outside of any matter. In our coordinates, the mass aspect evaluates to

$$m(r,t) = \frac{r\psi^6}{18\alpha^2}\left[r\beta' - \beta\right]^2 - 2r^2\psi'(r\psi)'. \tag{26}$$

Add a grid function to the RNPL code to compute the above expression. Varify that it behaves as expected, i.e. $m(r = 0) = 0$, $m(r)$ is strictly positive and monotonically increasing with r, and $m(r = r_{max})$ should grow *roughly* quadratically as a function of the amplitude of the initial data (why is this?). When the full evolution equations are complete we will verify that $m(r = r_{max})$ is conserved, modulo out-flux of scalar field energy.

Hints: When writing this routine it is strongly adviced to proceed in as many small steps as possible, testing statements (text and visual as in the sample program) will certainly prove helpful. Note that scalar field energy is the only source that can provide non-trivial solutions. However, having too large an initial energy density will prevent solution of the equations, so for testing first use zero scalar field to make sure you recover flatspace, then add add a small perturbation.

3.5 Problem 5. Solve the full Einstein-Klein-Gordon system

Implement (15) to evolve ψ at times $t > 0$. Add a flag to select between free versus constrained evolution for the conformal factor. Test that the code works by:

- Demonstrating 2nd order convergence.

- Show that you converge, 2nd order, to the same solution using free or constrained evolution. *NOTE: the boundary conditions are only consistent between the two methods to leading order in* $1/r$*, hence for a fixed outer boundary location convergence will fail after a certain point. Show that by increasing the outer boundary location you can maintain convergence to higher resolution.*

- Show that the mass of the spacetime is conserved, to second order, prior to any flux of scalar field energy leaving the domain.

Hints: use moderately weak-field data for these tests; i.e., choose initial data that changes the geometry noticeably, but does not lead to black hole formation.

3.6 Problem 6. Black hole formation

We will identify black hole formation with the formation of an *apparent horizon*, a surface whose *outgoing null expansion* θ is zero. One can think of the outgoing null expansion as follows. Place light sources all over the surface, and turn them on simultaneously at some time $t = \text{const}$. Two wavefronts will propagate away from the surface, one moving "outwards" (at least in a coordinate sense towards the asymptotically flat region of the spacetime), the other inwards. The fractional change in the area of the outward moving wavefront is θ—when this is zero, even though the wavefront is moving at the speed of light, it's not expanding at all. This is an apparent horizon, or *marginally outer trapped surface*. When apparent horizons form, the singularity theorems of Penrose and Hawking say a singularity must form interior to it as some time in the future. If *cosmic censorship* holds, then an event horizon will form that is always outside the apparent horizon. Inside the apparent horizon θ will become negative, i.e. the surfaces become *trapped*.

Due to the spherical symmetry of our problem, we know that any apparent horizon that forms must be a surface $r = \text{const}$. This allows for a simple derivation of the apparent horizon condition based directly on the description of the expansion given in the previous paragraph, as follows. In our metric, the areal radius R of an $r = \text{const}$ is $R = r\psi^2$, and the proper area is $A = 4\pi R^2$. The outgoing null expansion of this surface is then the fractional change of its area $A_{,\nu}/A$ in the direction of an outward pointing null vector ℓ^ν:

$$\theta \equiv (\ln A)_{,\nu}\, \ell^\nu = 2(\ln R)_{,\nu}\, \ell^\nu. \tag{27}$$

Being a null vector, there is some freedom in choosing the scale of ℓ^ν. We will fix the scale by constructing ℓ^ν as

$$\ell^\nu = n^\nu + s^\nu, \tag{28}$$

where n^ν is the usual unit timelike vector normal to $t = \text{const.}$ time slices, and s^ν is a radially outward-pointing *unit spatial vector*, i.e. $s_\nu s^\nu = 1$ and $n_\nu s^\nu = 0$. Verify that in our coordinates this uniquely determines s^ν to be:

$$s^\nu = \frac{1}{\psi^2}\left(\frac{\partial}{\partial r}\right)^\nu \tag{29}$$

Show that (27) evaluates to

$$\theta = \frac{2r}{3\alpha}\left(\frac{\beta}{r}\right)' + \frac{2}{r\psi^4}\left(r\psi^2\right)', \tag{30}$$

where (15) was used to simplify the expression. If there is an apparent horizon in the spacetime it will therefore coincide with the largest radius R where $\theta = 0$.

Implement a routine that searches for an apparent horizon, and when found prints the areal radius of the horizon. Consider a family of ingoing Gaussian initial data where the amplitude A_0 can be varied, but with fixed initial width δ and position $r_0 > \delta$. How does the initial energy of the spacetime scale with A_0? Based on this, what minimum amplitude would you expect to need for a black hole to form during the subsequent evolution of the field? Find initial data that causes a "modest" size black hole to form, i.e. one where the initial trapped surface encloses most of the scalar field.

Notes and hints: since we do not have excision implemented yet, we will not be able to evolve much past the point of apparent horizon formation before the code "crashes". In principle maximal slicing is so-called singularity avoiding, so should not run into a geometric singularity. However, as the singularity is approached, and without excision boundary conditions, the elliptic equations become ill-conditioned. This *may occur even before θ goes negative. A couple of "tricks" can be used to extend the evolution until one can find an apparent horizon. First, make sure you have adequate resolution in the regime where gravitational collapse is about to occur. Second, use free evolution for ψ. Third, try setting the* `correction_weight` *parameter, described in the sample elliptic source code, to* 0.5*; typically as the system begins to become ill-conditioned oscillations develop in the approximate solutions, and this parameter helps to smooth them out. Regarding the calculation of θ, it would be advantageous to normalize θ relative to its flat-space value; i.e. so that θ is* 1 *when evaluated in Minkowski spacetime.*

3.7 Problem 7. Sub-critical threshold behavior

Here we will begin to study critical phenomena at the *threshold* of black hole formation discovered by Choptuik in 1993. See Appendix B for a brief overview.

Since we do not yet have excision implemented, we will explore power-law scaling in the sub-critical regime. Also, we will not be able to get very close to threshold without adaptive mesh refinement, though we should be able to see the beginnings of the critical region. Using the Einstein equations, we relate the Ricci scalar $\mathcal{R}$ to the stress-energy tensor

$$\mathcal{R} = -8\pi T. \tag{31}$$

where $T = T_\nu{}^\nu$ is the trace of the stress tensor. Show that in our metric this evaluates to

$$\mathcal{R} = \frac{8\pi(\xi^2 - \Pi^2)}{\psi^4} \tag{32}$$

Add a grid function to the code that computes this. For the class of initial data considered in the previous question, use a bisection search to find the amplitude A_0^* corresponding to the threshold of black hole formation. For each *sub-critical* evolution record the maximum absolute value of $\mathcal{R}$ attained during evolution (there is no upper bound to $\mathcal{R}$ in the super-critical case, as $\mathcal{R}$ diverges at the singularity). From this data compute the critical scaling exponent γ (note that $\mathcal{R}$ has a dimension of $1/length^2$).

Note: given the ever smaller length-scales that become relevant approaching threshold, for any given resolution there will be a practical limit to how close to threshold you can reach. Once you are unable to reliably distinguish between black hole formation and dispersal stop the bisection search; the last reliable bracket will then give the uncertainty in A^* for that particular resolution. As usual, the Richardson expansion from several resolutions should be used to estimate the uncertainty in the numerically determined A^* relative to that of the desired analytical solution.

After Choptuik's discovery of critical phenomena, Stephen Hawking conceded, on a technicality, a bet that he had with Kip Thorne and John Preskill. Hawking believed naked singularities could not form from evolution of matter coupled to gravity if that matter exhibited no singular behavior by itself (i.e., in Minkowski spacetime with no back reaction). Extrapolating the scaling behavior to threshold, the critical solution is a naked singularity (i.e. R diverges in the limit, but there is no horizon in infinitesimally sub-critical evolution), and the scalar field of course has no singular behavior as a background field in Minkowski space beginning from regular initial data. The *technicality* Hawking claimed was that this naked singularity is a set of measure zero in all of parameter space, and so for any "practical" intents should not be considered a violation of cosmic censorship. However, *practically* a naked singularity would be a regime where curvatures reach the Planck scale and beyond, and this is *not* a set of measure zero for any family of initial data. Estimate how close to A^* you will need to fine-tune the initial data to reach Planck scale curvatures (or more specifically, so that the smallest length scale that unfolds is smaller than the Planck length L_p). Imagine an advanced civilization that can construct a focusing device powerful enough to form black holes from radiation that exhibits scaling behavior similar to the scalar field you have studied. Would the uncertainty principle allow them to fine-tune their initial data closely enough so that they could, on demand, form "Planck scale naked singularities"?

3.8 Time-permitting optional extensions

- Add excision and study black hole/non-linear scalar field interactions.

- Incorporate the code into the AMRD/PAMR frame work, solving the elliptic equations via multigrid. Compute the scalar field critical echoing exponent.
- Update the code to make it run in a parallel environment.

References

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A Solving 1-dimensional non-linear elliptic equations with a banded-matrix solver

Consider a differential equation with boundary conditions written as an operator $\mathcal{L}$ acting on a function $f(r)$, namely $\mathcal{L}[f(r)]$. Solutions to the differential operator are those functions $f(r)$ that satisfy

$$\mathcal{L}[f(r)] = 0. \tag{33}$$

We want to solve the discrete version of this, where we discretize the function f on a mesh of N points where $f_i = f(r = (i-1)h)$, $r \in 0..r_{max}$, $h = r_{max}/(N-1)$ and $i \in 1..N$. At each mesh point we replace the differential operator with a difference operator L, expanded about the given point using standard finite difference operators. We thus have converted the differential equation into a a set of N algebraic equations, one at each point on the mesh

$$L_i[f_k] = 0, \tag{34}$$

where in general the equation at point i can depend on the function values at all mesh points : $k \in 1..N$. Since we are interested in non-linear equations, we will solve this using Newton iteration. Start with a guess for the function, $\hat{f}_k$, so that

$$f_k = \hat{f}_k + \delta_k. \tag{35}$$

Substitute this into (34), and Taylor expand to first order in δ_k:

$$L_i[\hat{f}_k] + \sum_j J_{ij}[\hat{f}_k]\delta_j + O(\delta_k^2) = 0, \tag{36}$$

where we have introduced the *Jacobian*

$$J_{ij}[\hat{f}_k] = \frac{\partial L_i[\hat{f}_k]}{\hat{f}_j}. \tag{37}$$

$L_i[\hat{f}_k]$ is often called the *residual* $R_i[\hat{f}_k]$. To leading order then we just have a system of N linear equations for N unknowns δ_i, which we write in matrix form as

$$\mathbf{J}\vec{\delta} = -\vec{R}. \quad (38)$$

The solution is

$$\vec{\delta} = -\mathbf{J^{-1}}\vec{R}. \quad (39)$$

Inverting a large $N \times N$ matrix can be very expensive in general, requiring $O(N^3)$ operations. However here, because low-order finite different stencils only couple a small set of nearest neighbour points[1], the Jacobian will be a banded matrix with a small number of off-diagonal terms, which can be solved with $O(N)$ operations using a variant of Gaussian elimination. The `LAPACK` software library has efficient versions of this already implemented, and we will use the routine `DGBSV` to solve the above equations. An example code, `model_elliptic`, will be provided showing how the equation $f'' + 2f'/r - f^m = 0$, with $f'(r = 0) = 0$ and $f(r = r_{max}) = 1$ and m a constant, can be solved using `DGBSV`.

B Critical phenomena in gravitational collapse—a brief overview

Interesting and unexpected behavior occurs in self-gravitating systems at the *threshold* of black hole formation. This behavior, discovered by Choptuik in 1993 [1], was dubbed *critical phenomena*, in analogy to similar laws governing certain properties of statistical mechanical systems in phase transformations. Before describing the particular set of phenomena that occurs in gravitational collapse, it will be helpful to clarify the notion of the "threshold of formation" with the following canonical experiment.

Consider a smooth, compact distribution of scalar field energy at $t = t_0$, with an adjustable amplitude labeled by a parameter p. If we evolve this system, there are two possible end states as $t \to \infty$: for small initial amplitudes the scalar field will disperse to infinity, while sufficiently large amplitudes will induce gravitational collapse, leaving behind a black hole (with some of the scalar field escaping to infinity). This implies that there is a critical amplitude, corresponding to $p = p^\star$, such that for $p > p^\star$ black holes do form, while for $p < p^\star$ all of the scalar field escapes to infinity. $p = p^\star$ is the threshold of black hole formation, and it is in this vicinity of phase space that critical phenomena—*universality*, *scale invariance* and *power law scaling*—is seen. We will describe each of these phenomena in more detail below. Note, however, that the specific kind of critical behavior observed depends upon the matter model. Here, we focus on the massless scalar field—see [2] for a review article discussing the majority of systems studied to date, including a more extensive treatment of the massless scalar field than we are giving here.

One of the more remarkable aspects of a *critical solution*—the resulting geometry and matter field distribution in the limit as $p \to p^\star$ in the spacetime vicinity of the collapse— is its apparent universal nature. In other words, regardless of which particular one parameter (p) family of initial data is used to interpolate between dispersal and black hole formation[2], the resulting solution in the limit as $p \to p^\star$ is, up to certain trivial rescalings, *unique*. Moreover, in the vicinity of the critical solution in phase space ($|p - p^\star| \ll 1$), the deviation in the solution away from the critical one is uniquely specified by the number $p - p^\star$ (we explain this in more detail below).

Another of the features of the critical solution for the massless scalar field is that it is scale invariant. Specifically, the solution is discretely self-similar. Define a scale-invariant variable $x = r/(-t)$ and a time variable $\tau = -\ln(-t)$. Here r is a Schwarzschild-like (areal) radial coordinate and $t < 0$ is proper time as measured by a central observer, translated so that $t = 0$ coincides with the so-called *accumulation point*, which is the time when all length scales x within the solution collapse to zero proper length at $r = 0$. Then the discretely self-similar critical solution for a given variable (the scalar field, for instance) is a function $Z^\star(x, \tau)$ that is periodic in τ: $Z^\star(x, \tau) = Z^\star(x, \tau + \Delta)$. In (r, t) coordinates, one would see $Z^\star$ as a function oscillating in time, but with frequency diverging as $1/t$, and after each cycle, which is shorter that the previous one by a factor of e^Δ, the shape of the function repeats on a scale that is smaller by a factor e^Δ. Δ is called the *echoing exponent*.

[1] with standard second order operators only one on either side for interior points, and at most two toward the interior for boundary points where the boundary condition has derivatives of the function

[2] As long as the one parameter family is smooth in the vicinity of $p^\star$.

The critical solution is scale invariant, and thus has no intrinsic length scale. Near-critical solutions break this scale-invariance, introducing a length scale L into the solution that takes on the following form:

$$L \propto (p - p^\star)^\gamma. \tag{40}$$

For example, in a super-critical solution with $p > p^\star$ the mass M of the resulting black hole scales as

$$M \propto (p - p^\star)^\gamma. \tag{41}$$

Similarly, in a sub-critical solution with $p < p^\star$ the maximum value attained by the Ricci scalar (which has a dimension of $1/length^2$) during evolution is [3]

$$R_{max} \propto (p^\star - p)^{-2\gamma}. \tag{42}$$

This is the so called power law scaling behavior, and γ is called the *scaling exponent*. The scaling exponent is universal, although the constant of proportionality depends on the family of initial data, as well as the particular length scale considered.

From a dynamical systems point of view, the universal behavior can be explained if the critical solution is a *one-mode unstable* solution [4, 5]. This means that if one looks at linear perturbations about the critical solution $Z^\star$, and expands these perturbations as a sum of eigenfunctions with time dependence of the form $e^{\lambda\tau}$, then only a single eigenfunction will have $\lambda < 0$ and hence grow with time; the rest all decay with time. More specifically, suppose we have a near-critical solution with the following expansion:

$$Z(x,\tau;p) = Z^\star(x,\tau;p^\star) + \sum_i C_i(p - p^\star)e^{\lambda_i\tau}\xi_i(x,\tau), \tag{43}$$

where ξ_i is an eigenfunction periodic in τ, and each $C_i(p-p^\star)$ has the expansion $C_i(p-p^\star) = C_{i0}\cdot(p-p^\star) + O(p-p^\star)^2$ (so that $p = p^\star$ gives the critical solution). Let us label the growing mode λ_0, and suppose that at a time $\tau = \tau_0$ we specify initial data with some arbitrary perturbation, i.e., all C_i are in general non-zero. Then, as we evolve with time, all of the perturbations associated with decaying modes will eventually die out, leaving the following approximate solution at (say) $\tau = \tau_1$:

$$Z(x,\tau_1;p) - Z^\star(x,\tau_1;p^\star) \approx C_{00}\cdot(p - p^\star)e^{\lambda_0\tau_1}\xi_0(x,\tau_1), \tag{44}$$

where we have assumed that $p - p^\star$ is small. This explains the universality of the critical and near-critical solutions. Furthermore, we can obtain an explicit relationship between λ_0 and γ using the following argument. The perturbed critical solution posited in (43) will persist in time until the right hand side of (44) has grown to a value of order unity, at say $\tau = \tau_L$. After that, one would expect the non-linear terms to push the solution to either dispersal or black hole formation. However, at $\tau = \tau_L$, the only length-scale that has developed in the solution is τ_L, and hence this is the only scale that the subsequent solution could inherit. This gives

$$C_{00}(p - p^\star)e^{\lambda_0\tau_L}\xi_0(x,\tau_L) \approx 1 \quad \Rightarrow \tag{45}$$
$$\lambda_0\tau_L = \ln(p - p^\star) + \text{const}, \quad \Rightarrow \tag{46}$$
$$\tau_L \propto \ln(p - p^\star)^{1/\lambda_0}. \tag{47}$$

Comparing the last line above with (40), we get that $\gamma = 1/\lambda_0$. This relationship has been verified in perturbation theory in [6, 7]. Note however, that we have been a little bit sloppy in the above derivation; in particular, the fact that $\xi_0(x,\tau)$ is periodic in τ introduces a secondary scale into the problem, whose value depends upon when in the cycle of ξ_0's oscillation the amplitude of the mode grows above unity. This is a minor effect, but nevertheless introduces a small, periodic "wiggle" (with period $\Delta/(2\gamma)$) on top of the linear relationship $n\ln(L) = \gamma(p - p^\star)$, for a length scale of L^n [6, 8].